

# ON QUANTUM GROUP $GL_{p,q}(2)$

TANYA KHOVANOVA

January 28, 1997

ABSTRACT. In the Hopf algebra  $Gl_{p,q}(2)$  the determinant is central iff  $p = q$ . In this case we put determinant to be equal to 1 to get  $SL_q(2)$ . In this paper we consider the case when  $p/q$  is a root of unity; and, consequently, a power of the determinant is central.

## Introduction.

In the paper [B-Kh] the Universal enveloping algebras for metaplectic quantum groups of  $SL(2)$ -type were constructed. As we look back at the history of  $SL_q(2)$  we can notice that on the levels of formulas the algebra of functions on  $SL_q(2)$  was always presented the same way ([M], [R-T-F]), while the Universal enveloping algebra had different presentations. This fact encouraged me and Joseph Bernstein [B-Kh] to consider the enveloping algebra as the secondary object in comparison with the Hopf algebra of functions. As a byproduct of our axiomatic search we constructed the metaplectic quantum groups of  $SL(2)$ -type. This construction was carried out on the level of universal algebras.

On my way to find nice formulas for algebra of functions on metaplectic quantum group, I found some nice formulas for other algebras. These formulas are presented in this paper.

I am thankful to Joseph Bernstein for his interest in this subject and helpful discussions.

## 1. Algebra of regular functions on $GL_{p,q}(2)$ .

**1.1.** For every  $p, q \in \mathbb{C}^*$  we consider the bialgebra  $Mat_{p,q}$  which is generated by four noncommuting elements  $(a, b, c, d)$ , satisfying the following relations (see, [M], [O-W] and references there):

$$\begin{aligned} ab &= p^{-1}ba & ac &= q^{-1}ca \\ cd &= p^{-1}dc & bd &= q^{-1}db \\ bc &= q^{-1}pcb & ad - da &= (p^{-1} - q)bc. \end{aligned} \tag{*}$$

Introduce matrices

$$Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Mat}(2, \text{Mat}_{p,q})$$

$$P = \begin{pmatrix} 0 & -1 \\ p^{-1} & 0 \end{pmatrix} \quad Q = \begin{pmatrix} 0 & -1 \\ q^{-1} & 0 \end{pmatrix}.$$

Then we can rewrite the relations (\*) in a more compact form:

$$Y P Y^t = D P$$

$$Y^t Q Y = D Q,$$

where  $D$  is an element in  $\text{Mat}_{p,q}$  and has a meaning of quantum determinant.

*Comment.* The matrix  $P$  defines a quantum plane as an algebra generated by two generators  $x, y$  and the relation  $xy = p^{-1}yx$ . Analogously the matrix  $Q$  defines a quantum plane. We consider  $\text{Mat}_{p,q}$  as an algebra of operators preserving quantum plane defined by  $P$  and dual quantum plane defined by  $Q$ .

**1.2.** The comultiplication in the algebra  $\text{Mat}_{p,q}$  is defined as follows:

$$\begin{aligned} \Delta a &= a \otimes a + b \otimes c \\ \Delta b &= a \otimes b + b \otimes d \\ \Delta c &= c \otimes a + d \otimes c \\ \Delta d &= c \otimes b + d \otimes d. \end{aligned} \tag{**}$$

Using the natural imbeddings  $i', i'' : \text{Mat}_{p,q} \rightarrow \text{Mat}_{p,q} \otimes \text{Mat}_{p,q}$ , ( $i'(x) = x \otimes 1$ ,  $i''(x) = 1 \otimes x$ ), we can rewrite comultiplication formulae (\*\*) as follows:

$$\Delta(Y) = i'(Y) \cdot i''(Y),$$

which is an equality in  $\text{Mat}(2, \text{Mat}_{p,q} \otimes \text{Mat}_{p,q})$ .

**1.3.** This algebra has a multiplicative quantum determinant  $D = \det_{p,q}(Y)$  (see [M], [O-W], [K]):

$$D = da - pcb = da - qbc = ad - p^{-1}bc = ad - q^{-1}cb.$$

Multiplicativity means that  $\Delta D = D \otimes D$ , or, equivalently,  $\det_{p,q}(Y_1 Y_2) = \det_{p,q}(Y_1) \cdot \det_{p,q}(Y_2)$  whenever the entries of  $Y_1$  commute with the entries of  $Y_2$ .

Localizing by  $D^{-1}$  we will get the algebra of functions on  $GL_{p,q}(2)$ . This localization can be described easily due to the fact that  $D$  is normalizing (see [M]):

$$\begin{aligned} Da &= aD & Db &= p^{-1}qbD \\ Dc &= pq^{-1}cD & Dd &= dD. \end{aligned}$$

**1.4.** The bialgebra  $\text{Fun}(GL_{p,q})$  is the Hopf algebra; an antipode  $S$  is defined by:  $S(Y) = Y^{-1}$ . Specifically:

$$\begin{aligned} S(a) &= dD^{-1} & S(b) &= -pbD^{-1} \\ S(c) &= -p^{-1}cD^{-1} & S(d) &= aD^{-1}. \end{aligned}$$

In a more compact form:

$$Y^{-1} = S(Y) = PY^tP^{-1}D^{-1} = D^{-1}Q^{-1}Y^tQ.$$

**1.5.** If  $p = q$  then  $D$  is central; and we can take quotient of  $Fun(GL_{p,q}(2))$  by the Hopf ideal generated by  $D - 1$ . We would get standard  $Fun(SL_q(2))$  (see [Kas]).

**1.6.** Suppose now that  $p^{-1}q$  equals  $\xi$ , where  $\xi$  is the  $n$ -th root of unity:  $\xi^n = 1$ . In this case  $D^n$  is central. An ideal generated by  $D^n - 1$  is a Hopf ideal, so we can consider a quotient algebra, which we would denote by  $Fun(SL_{q,\xi}(2))$ :

$$Fun(SL_{q,\xi}(2)) = Fun(GL_{p,q}(2))/(D^n - 1).$$

## 2. Axiomatic approach.

**2.1.** Analyzing the Hopf algebra  $A = Fun(SL_{q,\xi}(2))$  we note that it has the following important property:

Let  $I \subset A$  be a two-sided ideal generated by  $b$  and  $c$ . Then  $I$  is a Hopf ideal in  $A$ , i.e.  $\Delta I \in A \otimes I + I \otimes A$  and  $S(I) \subset I$ . The quotient Hopf algebra  $A/I$  is isomorphic to the algebra of functions on the algebraic group  $\mathbb{C}^* \otimes \mathbb{Z}_n$ :

$$\mathbb{C}[a, d, D, D^{-1}]/(ad - D, D^n - 1)$$

$$\Delta a = a \otimes a \quad \Delta d = d \otimes d \quad \Delta D = D \otimes D$$

$$S(a) = dD^{-1} \quad S(d) = aD^{-1} \quad S(D) = D^{-1}.$$

Or, equivalently:

$$\mathbb{C}[a, a^{-1}, D, D^{-1}]/(D^n - 1)$$

$$\Delta a = a \otimes a \quad \Delta D = D \otimes D$$

$$S(a) = a^{-1} \quad S(D) = D^{-1}.$$

Informally, this means that our quantum group  $A = Fun(SL_{q,\xi}(2))$  contains the group  $\mathbb{C}^* \otimes \mathbb{Z}_n$  as a subgroup.

## 3. Dual picture.

**3.1.** Let us denote by  $T^2$  the two-dimensional torus. Denote by  $t$  a point of  $T^2$ . To each point  $t = (x_1, x_2)$  we can correspond a generator  $\hat{t}$  in the group algebra of torus. This correspondence is multiplicative —  $\widehat{t_1 t_2} = \hat{t}_1 \hat{t}_2$ .

We can describe (we use notations similar to [B-Kh]) a universal enveloping algebra  $U_{p,q}(2)$  of  $GL_{p,q}(2)$  as a Hopf algebra generated by  $\hat{t}$  ( $t \in T^2$ ) and two elements  $E, F$ , satisfying the relations:

$$\begin{aligned} \hat{t}E\hat{t}^{-1} &= \alpha(t)E \\ \hat{t}F\hat{t}^{-1} &= (-\alpha)(t)F \end{aligned} \tag{1}$$

$$\begin{aligned}
\Delta \hat{t} &= \hat{t} \otimes \hat{t} \\
\Delta E &= E \otimes 1 + Q_1 \otimes E \\
\Delta F &= F \otimes Q_2^{-1} + 1 \otimes F
\end{aligned} \tag{2}$$

$$(3) \quad [E, F] = \frac{Q_1 - Q_2^{-1}}{q - p^{-1}}.$$

Here  $Q_1, Q_2$  are generators corresponding to points of the torus with coordinates  $(q, p^{-1})$  and  $(p, q^{-1})$  respectively; and  $\alpha$  denotes the weight  $(1, -1)$  on torus: if  $t = (x, y)$ , then  $\alpha(t) = xy^{-1}$ .

*Comments.* 1. The commutator  $[E, F]$  is the element  $X$  of an algebra, generated by torus, satisfying the equation:  $\Delta X = Q_1 \otimes X + X \otimes Q_2^{-1}$ . In denominator (3) we can choose any constant. We chose our constant so that the evaluation of the right hand side element on the weight  $(1, 0)$  would be equal to 1.

2. In this case and in the following cases, the antipode is uniquely defined and could be easily recovered:

$$S(\hat{t}) = \hat{t}^{-1} \quad S(E) = -Q_1^{-1}E \quad S(F) = -Q_2F.$$

**3.2.** To get the universal algebra  $U_q(2)$  of  $SL_q(2)$  we have to put  $p = q$  and to take the group subalgebra of  $T^1 \subset T^2$  generated by elements  $\hat{h}$  corresponding to  $(h, h^{-1}) \in T^2$ . Denote by  $K$  an element corresponding to  $(q, q^{-1})$ , then we would have the following relations:

$$\begin{aligned}
(1) \quad \hat{h}E\hat{h}^{-1} &= h^2E \\
\hat{h}F\hat{h}^{-1} &= h^{-2}F
\end{aligned}$$

$$\begin{aligned}
(2) \quad \Delta \hat{h} &= \hat{h} \otimes \hat{h} \\
\Delta E &= E \otimes 1 + K \otimes E \\
\Delta F &= F \otimes K^{-1} + 1 \otimes F
\end{aligned}$$

$$(3) \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

**3.3.** Consider the case  $SL_{q,\xi}(2)$ : namely,  $p^{-1}q = \xi$ , where  $\xi^n = 1$ . We can describe the universal enveloping algebra  $U_{q,\xi}(2)$  as a subalgebra in  $U_{p,q}(2)$ , generated by elements  $E, F, \hat{h} = (h, h^{-1}) \in T^2$  for  $h \in T$  and  $W = (1, \xi) \in T^2$  ( $W^n = 1$ ).

$$\begin{aligned}
(1) \quad WEW^{-1} &= \xi^{-1}E & \hat{h}E\hat{h}^{-1} &= h^2E \\
FWF^{-1} &= \xi F & \hat{h}F\hat{h}^{-1} &= h^{-2}F
\end{aligned}$$

$$\begin{aligned}
(2) \quad \Delta W &= W \otimes W \\
\Delta \hat{h} &= \hat{h} \otimes \hat{h} \\
\Delta E &= E \otimes 1 + W\hat{q} \otimes E \\
\Delta F &= F \otimes W\hat{\xi}\hat{q}^{-1} + 1 \otimes F
\end{aligned}$$

$$(3) \quad [E, F] = \frac{\hat{q} - \hat{\xi}\hat{q}^{-1}}{q - \xi q^{-1}} W.$$

## References

[B-Kh] J.Bernstein and T.Khovanova, *On quantum group  $SL_q(2)$* , Comm. Math. Phys., **117** (1996), 691-708. *hep-th/9412056*.

[K] B.A.Kupershmidt, *Classification of quantum group structures on the group  $GL(2)$* , J. Phys. A., **27** (1994), L47-L51.

[Kas] Ch.Kassel, *Quantum groups*, Springer-Verlag. 1995.

[M] Yu.I.Manin, *Topics in Noncommutative Geometry*, Princeton University Press, 1991.

[O-W] O.Ogievetsky, J.Wess, *Relations between  $GL_{p,q}(2)$ 's*, Z. Phys. C, **50** (1991), 123-131.

[R-T-F] N.Yu.Reshetikhin, L.A.Takhtadzhyan and L.D.Faddeev, *Quantization of Lie groups and Lie algebras*, Leningrad Math. J. **1** (1990), 193-225.

*Present address:* TANYA KHOVANOVA, DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, NJ 08544

*E-mail address:* tanya@math.princeton.edu